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COMPARISON OF HALF INTEGER AND THIRD INTEGER EXTRACTION FOR THE ENERGY DOUBLER

1. BASIC PROCESSES

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I. Introduction

Discussions of slow extraction in the various Energy Doubler reports over the years[1] have emphasized the third-integer process. This preference is not surprising, for the analytical treatment of third-integer extraction is quite compact and only a few paragraphs are needed to make estimates of the parameters. One would not expect dramatic differences in cost or space allocation to hinge on the choice of resonance.

However, the Doubler design is now at the level of detail where a closer look at the process is appropriate. These notes present a discussion of the elementary third-integer and half-integer extraction resonances. Insofar as possible, the same approach is used in treating the two cases in the hope of making differences and relative advantages clear. In this memorandum, no attempt is made at refinement of either process.

Whatever extraction resonance is used, the accelerator imposes some constraints common to all. The aperture is limited, and there is a maximum oscillation amplitude that one would not wish to exceed. We will call this limiting amplitude x_{max}. We will assume that the displacement at a few places may be allowed to exceed x_{max} by using "special measures" (larger aperture magnets, hope, etc.). The positions of the extraction septa are examples of such places.

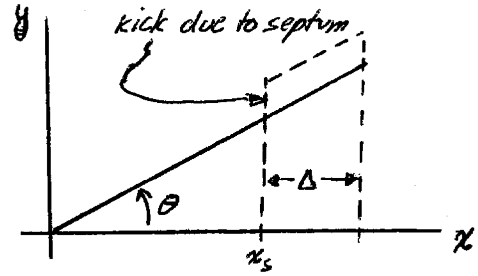
Throughout this paper, x_s will denote the distance by which the inner edge of the septum channel (i.e., the wires of the electrostatic septum) is offset from the central trajectory, and Δ will stand for the septum aperture. Of course, Δ isn't a constant of nature, but given the effort that has been devoted to electrostatic septum design over the last ten years, one does not change this fugure lightly.

The same symbol, Δ, will also denote the step size. In general, there is little benefit to be derived from extensive discussion of the cases where the step size and septum aperture are not the same. However, if the two need be distinguished, suitable remarks will be made.

The three quantities mentioned above are related by

$$x_s + \Delta = x_{max}\left(\frac{\beta_s}{\beta_o}\right)^{1/2}\cos\theta$$

where β_s is the amplitude function at the septum, β_o is the amplitude function at points consistent with x_{max}, and θ characterizes the phase of the oscillation at the electrostatic septum. The sketch at the right illustrates the phase space at the electrostatic septum. We use coordinates where x is the displacement in the extraction plane (assumed to be the horizontal plane) and $y = \beta_s x' + \alpha_s x$.



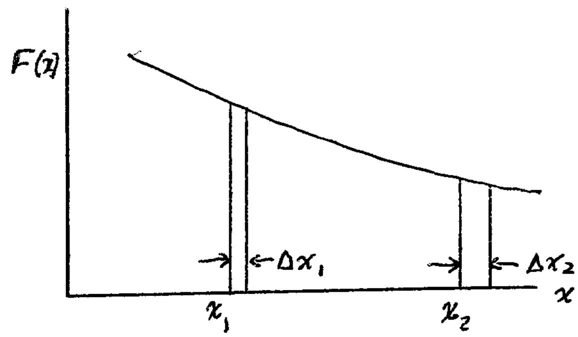
At this writing, it appears that the electrostatic septum will be located at the upstream end of the F straight section. As in the main ring, a magnetic septum is located in straight section A. A lattice modification that produces amplitude functions favorable to extraction at both septa has been devised by Collins.[2] The advantages of this modification of β have been outlined elsewhere.[3] We will assume that this lattice modification is adopted, and take $\beta_s = 225$ m, in contrast to $\beta_o = 100$ m.

Comments on the phase angle θ will be made below. the tolerable amplitude x_{max} has been examined by Collins[4] and by H. Edwards and Harrison[5]; following their conclusions, we will take x_{max}=20mm.

II. Inefficiency

In order to estimate extraction inefficiency, let us suppose that extraction proceeds so slowly that it may be considered a static process. Then, the particle density distribution along an outgoing separatrix or along the projection of the separatrix onto a coordinate axis varies inversely as the rate of change of position along that coordinate. To convince oneself of this, let $F(x_1)\,\Delta x_1$ be the number of particles in an interval Δx_1 at x_1. After some time interval T has elapsed, the particles find themselves in Δx_2 at x_2. The number of particles is the same, so

$$F(x_2)\Delta x_2 = F(x_1)\Delta x_1$$



From

$$T = \int_{x_1}^{x_2} \frac{dx}{(dx/dt)} = \int_{x_1+\Delta x_1}^{x_2+\Delta x_2} \frac{dx}{(dx/dt)}$$

it follows that

$$\int_{x_1}^{x_1+\Delta x_1} \frac{dx}{(dx/dt)} = \int_{x_2}^{x_2+\Delta x_2} \frac{dx}{(dx/dt)}$$

or

$$\frac{\Delta x_1}{(dx/dt)_1} = \frac{\Delta x_2}{(dx/dt)_2}$$

So Δx varies directly as dx/dt; therefore F(x) varies inversely as dx/dt. It will be more convenient to use the "turn number", n, as the independent variable; that is, we take the spatial dependence of F to be of the form

$$F \propto \frac{1}{(dx/dn)}$$

If a septum of thickness w in the x-coordinate is located at a distance x_s from the central orbit, then the inefficiency, ϵ, defined as the fraction of the particles that strike the septum is

$$\epsilon = \frac{\int_{x_s}^{x_s+w} dx/(dx/dn)}{\int_{x_s}^{\infty} dx/(dx/dn)} = \frac{\int_{x_s}^{x_s+w} dx/(dx/dn)}{\int_{x_s}^{x_s+\Delta} dx/(dx/dn)}$$

The second form above acknowledges, in the denominator, that the particle density distribution cuts off at a distance $x_s + \Delta$, Δ being the "step-size", the growth in x in the number of turns N between successive encounters with the septum at the proper phase for exit from the machine. For half-integer extraction, N=2; in the third-integer case, N=3.

The septum thickness, w, is small compared to x_s, and the integral in the numerator can be replaced by w/(dx/dn) evaluated at x_s. The integral in the denominator is just N, and sometimes it will be convenient to identify it as such. Thus, for the inefficiency, we will use either of the following forms:

$$\epsilon = \frac{w}{(dx/dn)_{x_s}} \frac{1}{\int_{x_s}^{x_s+\Delta} dx/(dx/dn)} = \frac{1}{N} \frac{w}{(dx/dn)_{x_s}}$$

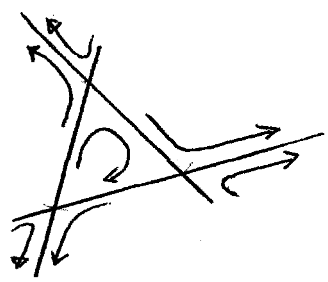
Note that the flatter the distribution F, the better the efficiency. Other considerations aside for the moment, this circumstance favors the choice of low-order multipoles to generate the step-size - the lower the multipole order, the less steep the dependence of dx/dn on x. As a result, in the discussion of half-integer extraction, it should not be surprising that we will concentrate on the situation where the quadrupoles rather than the octupoles dominate in the rate of growth of the unstable oscillations.

III. Third-Integer Extraction

We take up the third-integer case first. As noted in the introduction, the equations are simpler and less numerous than in the half-integer version. Also, for third-integer, the zero stable phase space limit is clear and can be treated before taking up the finite stable phase space situation. For both resonances, the essentials are illustrated by the zero stable phase space limit, but for half-integer that limit is neither apparent at the outset, nor particularly helpful when identified.

The equations of motion will be stated without proof in the body of this report. They are reasonably familiar, especially for third-integer, and can be obtained in a variety of ways. But, for completeness, derivations are presented in the Appendix.

The design tune of the Energy Doubler is $\sim$19.4 in both planes of oscillation. To initiate third-integer extraction, the horizontal tune is shifted toward the resonant value of 19+1/3, and a 3x(19+1/3) harmonic of sextupole is turned on to produce the requisite partition of phase space into stable and unstable regions. We need not go into here precisely how this 58th harmonic is set up around the ring; numerical integration of the equations of motion shows that the process is surprisingly insensitive to the details of the sextupole distribution. In the x,y coordinates defined in the introduction, the figure that marks the boundary between stable and unstable regions - the separatrix - is an equilateral triangle. The flow of particles in phase space is shown in the sketch. We will call the three extensions of the triangle sides along which particles stream with increasing amplitude the "outgoing separatrices".



As the tune is gradually reduced toward 19+1/3, particles are squeezed out of the ever-smaller stable region at the corners of the triangle; once outside, they proceed along the outgoing separatrices. For a tune value arbitrarily close to 19+1/3, the stable area becomes infinitesimal and particles flow out from the origin on three lines inclined at angles of 120° from one another. We treat this case first.

A. On Resonance (ν = 19+1/3)

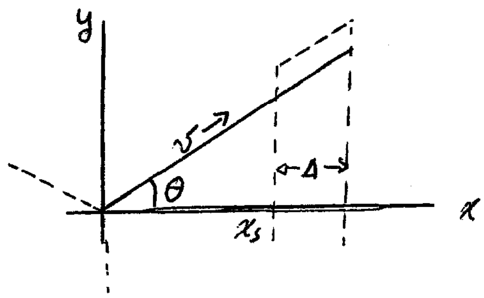
The equation of motion is

$$\frac{dv}{dn} = \frac{1}{4} S v^2 \tag{1}$$

where v is the distance along an outgoing separatrix, and

$$S \equiv -\frac{\beta_s}{B\rho}\oint\left(\frac{\beta(z)}{\beta_s}\right)^{3/2}\left(\frac{B''}{2}\right)\cos(58\varphi+3\theta)\,dz \tag{2}$$

is the driving term due to the sextupole distribution around the ring. As usual, $\varphi \equiv \int dz/(\nu\beta)$; $\varphi = 0$ at the electrostatic septum. The product $B\rho$ is the magnetic rigidity. If S is produced by sextupoles of length L located at β_0 in the standard cells, then



$$S = -\frac{\beta_0^{3/2}}{\beta_s^{1/2}}\frac{1}{B\rho}\sum_i\left(\frac{B''L}{2}\right)_i\cos(58\varphi_i+3\theta) \tag{3}$$

Projected on the x-axis, the equation of motion becomes

$$\frac{dx}{dn} = \frac{1}{4}\frac{S}{\cos\theta}x^2 \tag{4}$$

If, in 3 turns, a particle initially at x_s is to progress to $(x_s+\Delta)$, integration of the equation of motion gives

$$\frac{1}{x_s} - \frac{1}{x_s+\Delta} = \frac{3S}{4\cos\theta} \tag{5}$$

For the inefficiency, the formula of Section II yields

$$\epsilon = w\left(\frac{1}{x_s}+\frac{1}{\Delta}\right) = w\left(\frac{x_s+\Delta}{x_s\Delta}\right) \tag{6}$$

Now recall that $x_s+\Delta$ is related to x_{max} by

$$x_s+\Delta = x_{max}\left(\frac{\beta_s}{\beta_0}\right)^{1/2}\cos\theta \tag{7}$$

and so for fixed values of the quantities on the right hand side of (7), the value of $x_s+\Delta$ is determined. The minimum inefficiency occurs for $x_s = \Delta$:

$$\epsilon_{min} = \frac{4w}{(x_s+\Delta)} \tag{8}$$

For best efficiency, one would like θ to be as small as possible; an idea of how small it can be may be obtained by rotating the diagram above by 80° to see how the picture looks at the magnetic septum. Then the kick from the electrostatic septum is almost fully projected on the x-axis regardless of θ, but for $\theta = 0^\circ$ the magnetic septum would be in the middle of the aperture and even if the stable phase space were truly negligible, would shadow one of the other outgoing separatrices shown as dotted lines. Probably θ can be no smaller than 25-30°: it may have to larger due to aperture limitations.

As a numerical example, let us take

$$x_{max} = 20 \text{ mm}$$
$$\beta_s = 225 \text{ m}$$
$$\beta_o = 100 \text{ m}$$
$$\theta = 30^\circ$$
$$w = 0.004'' = 0.1 \text{ mm}$$

Then

$$x_s+\Delta = 26 \text{ mm}$$
$$\epsilon_{min} = 1.54\%$$
$$\epsilon = 1.63\% \text{ for } \Delta = 10 \text{ mm.}$$

Note that the change in the inefficiency between the minimum and that which is consistent with the existing electrostatic septum design is negligible.

The step size condition (5) can be used to find S. A more useful form is obtained by evaluating S with β_s replaced by β_o. Call the sextupole term so defined S_o. Also let

$$f \equiv \frac{x_s}{x_s+\Delta} \tag{9}$$

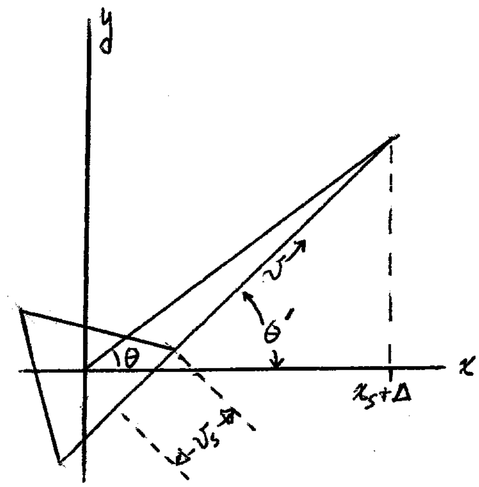
Then (5) becomes

$$\frac{1}{f} - 1 = \frac{3}{4} S_o x_{max} \tag{10}$$

B. Finite Stable Phase Space

Let δ be the difference between the tune and 19+1/3; i. e.,

$$\nu = 19\tfrac{1}{3} + \delta \tag{11}$$



The equation of motion along an outgoing separatrix may be written

$$\frac{dv}{dn} = \frac{1}{4} S\left(v^2 - v_s^2\right) \tag{12}$$

with

$$v_s = \sqrt{3}\left(\frac{4\pi\delta}{S}\right) \tag{13}$$

The sextupole term S is defined as in (2), except that θ is replaced by θ'.

In order to preserve the relationship between $x_s + \Delta$ and x_{max}, the separatrix has been rotated slightly and is inclined at an angle θ' with respect to the x-axis. The angles θ and θ' are related by

$$\tan\theta' - \tan\theta = \frac{v_s/\sqrt{3}}{(x_s+\Delta)} = \frac{v_0/\sqrt{3}}{x_{max}\cos\theta} \tag{14}$$

In the last form of (14), v_0 corresponds to v_s evaluated at β_0.

The area of the triangle is determined by the emittance $\mathcal{E}$ of the stable beam:

$$v_0^{\,2} = \frac{\beta_0\,\mathcal{E}}{\sqrt{3}} \tag{15}$$

In order to integrate the equation of motion to find the step size condition, project v on the x-axis using

$$x = v\cos\theta' + \frac{v_s}{\sqrt{3}}\sin\theta' \tag{16}$$

Then upon integration the step size condition is

$$\ln\left[\frac{1-f_a}{1+f_b}\ \frac{f+f_b}{f-f_a}\right] = \frac{3}{2} v_0 S_0 \tag{17}$$

where, as before, $f = x_s/(x_s+\Delta)$, and

$$\begin{aligned} f_a &= \frac{v_0}{x_{max}\cos\theta}\left[\cos\theta' + \frac{1}{\sqrt{3}}\sin\theta'\right] \\ f_b &= \frac{v_0}{x_{max}\cos\theta}\left[\cos\theta' - \frac{1}{\sqrt{3}}\sin\theta'\right] \end{aligned} \tag{18}$$

For given f, (15) and (18) are conditions to determine S_o and δ.

In terms of the same parameters, the inefficiency becomes

$$\epsilon = \epsilon_{min}\frac{1}{2}\frac{v_0}{x_{max}}\ \frac{\cos\theta'}{\cos\theta}\frac{1}{(f-f_a)(f+f_b)}\ \frac{1}{\ln\left[\frac{1-f_a}{1+f_b}\ \frac{f+f_b}{f-f_a}\right]} \tag{19}$$

with ϵ_{min} given by (8).

As an example to compare with the vanishing stable phase space case, take $\mathcal{E} = 0.04\pi$ mm-mrad as might be the case in the 400 GeV range. As before, $\theta = 30^\circ$ and $\beta_0 = 100$ m. Then

from (15) $v_o = 2.69$ mm
from (14) $\theta' = 33.71^\circ$
from (18) $f_a = 0.1792$
$f_b = 0.0795$

For f = 0.6:

from (17) $S_o = 0.051$ mm^{-1}
from (13) $\delta = 0.0063$
from (19) $\epsilon = 1.10\,\epsilon_{min}$

Of course, (13) can just as well be written $v_o = \sqrt{3}(4\pi\delta/S_o)$, and was used in the latter form to obtain δ above.

The transition to finite stable phase space has not resulted in any substantial differences from the zero phase space case. The inefficiency is a bit higher and the sextupole strength has risen by some 15%.

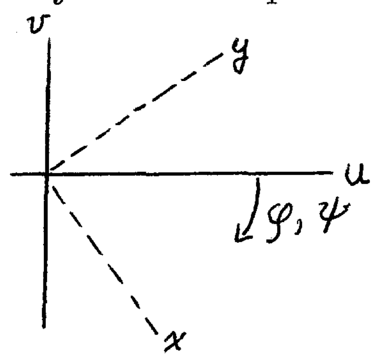
The behavior of the inefficiency as a function of f is illustrated for a few cases in Figure 1.

IV. Half-Integer Extraction

The nearest half-integer to the operating tune is at $\nu = 19\frac{1}{2}$. A 39th quadrupole harmonic will create a stop-band at $19\frac{1}{2}$. With quadrupoles alone, however, all particles are either stable or unstable; a non-linear element is needed to separate the phase space into stable and unstable regions. A zero harmonic octupole term yields a dependence of tune on amplitude, and so could satisfy the need; the main ring extraction system uses this approach. The outgoing separatrices are curved; they are, in fact, arcs of circles. A 39th harmonic of octupole can also perform the separation, and, as will be seen, provides outgoing separatrices which become straight for sufficiently large amplitude. We will examine the latter option here.

A. Equations of Motion and Selection of Parameters

In order to avoid excessive complexity of the expressions which follow, we will write them in terms of axes u, v which are related to x, y by a rotation to be defined below. As in the third-integer case, the equations of motion can be obtained by any of the usual techniques, one of which is presented in the Appendix. We state first the rate of change of amplitude $a = (u^2+v^2)^{\frac{1}{2}}$:



$$\frac{da}{dn} = \frac{1}{2} a Q \sin 2\psi + \frac{1}{4} a^3 E \sin 2\psi \tag{1}$$

where Q, E - the quadrupole and octupole terms respectively - are defined by

$$Q \equiv \frac{\beta_s}{B\rho} \oint \left(\frac{\beta(z)}{\beta_s}\right) B' \cos(39\varphi)\, dz$$

$$E \equiv \frac{\beta_s}{B\rho} \oint \left(\frac{\beta(z)}{\beta_s}\right)^2 \left(\frac{B'''}{6}\right) \cos(39\varphi)\, dz \tag{2}$$

In (1), we can recognize the relative contribution to amplitude growth from the quadrupole and octupole terms:

$$\frac{\text{growth from quadrupole}}{\text{growth from octupole}} = \frac{2Q}{Ea^2}$$

From the discussion of inefficiency in Section II, we already expect that we want the quadrupoles to dominate in producing step size. One way of expressing this dominance is the ratio above evaluated at x_{max}; that is, we define a quantity R by

$$R = 2Q_o/(E_o x_{max}^{\ 2}) \tag{3}$$

where, as before, the subscripts indicate the terms are evaluated at β_o ; i.e., replace β_s by β_o in (2). (Note Q_o=Q.)

In u, v coordinates, the equations of motion are

$$\frac{du}{dn} = -\frac{1}{2}Qv + 2\pi\delta v - \frac{1}{2}Ev^3 \tag{4}$$

$$\frac{dv}{dn} = -\frac{1}{2}Qu - 2\pi\delta u - \frac{1}{2}Eu^3 \tag{5}$$

with,

$$\delta \equiv \nu - 19\tfrac{1}{2} \tag{6}$$

Observe that $\delta < 0$, since we are approaching the resonance "from below".

Look at the small amplitude motion first - we want it to be stable. If we neglect the cubic terms and divide (4) by (5), we have

$$\frac{du}{dv} = \frac{Q-4\pi\delta}{Q+4\pi\delta}$$

or

$$u^2 = \left(\frac{Q - 4\pi\delta}{Q + 4\pi\delta}\right) v^2 + \text{constant} \tag{7}$$

For the trajectories in u, v phase space to be closed, we must have

$$\left(\frac{Q - 4\pi\delta}{Q + 4\pi\delta}\right) < 0$$

or

$$Q^2 < (4\pi\delta)^2 \tag{8}$$

This inequality is not surprising; $Q/4\pi\delta$ is the half-width of the stop-band caused by Q, and one would not wish the tune difference δ to be less in magnitude than the half-width of the stop-band. The ratio $Q/4\pi\delta$ thus has physical significance and we will use it as a parameter. Let us define k by

$$k \equiv -\frac{Q}{4\pi\delta} \; ; \quad 0 \leq k \leq 1 \tag{9}$$

where the negative sign is inserted in anticipation of the discussion below where we will take $Q > 0$.

Next, find the fixed points; i.e., the points where du/dn = dv/dn = 0.

$$\frac{du}{dn} = 0 \Rightarrow v = 0 \quad \text{or} \quad v^2 = -\frac{(Q - 4\pi\delta)}{E}$$

$$\frac{dv}{dn} = 0 \Rightarrow u = 0 \quad \text{or} \quad u^2 = -\frac{(Q + 4\pi\delta)}{E}$$

The choice u = v = 0 isn't interesting - that's just the central trajectory. Nor is the case where both u and v are non-zero. For the product of the two expressions on the right above is

$$u^2 v^2 = \frac{Q^2 - (4\pi\delta)^2}{E^2}$$

which, according to (8), is less than zero, so u and v cannot both be real.

The remaining two choices - u=0, v≠0 or v=0, u≠0 - just differ in the signs of Q and E; the physical content is the same. (The equations of motion are invariant under the changes Q'=-Q, E'=-E, v'=-u, u'=v.) We arbitrarily select the case for which Q and E are positive.

$$v = 0; \quad u_s^2 = -\frac{(Q + 4\pi\delta)}{E}$$
$$\text{or} \quad u_0^2 = -\frac{(Q_0 + 4\pi\delta)}{E_0} \tag{10}$$

Let $U = u/u_s$; $V = v/u_s$. In terms of U, V the equations of motion become

$$\frac{1}{1-k}\,\frac{1}{2\pi\delta}\,\frac{dU}{dn} = \frac{1+k}{1-k}V + V^3 \tag{11}$$

$$\frac{1}{1-k}\,\frac{1}{2\pi\delta}\,\frac{dV}{dn} = -U + U^3 \tag{12}$$

from which a first integral is

$$-\frac{U^2}{2} + \frac{U^4}{4} = \frac{1}{2}\left(\frac{1+k}{1-k}\right)V^2 + \frac{V^4}{4} + \text{constant} \tag{13}$$

At the fixed point, U=1, V=0, and the value of the constant is $-\frac{1}{4}$. The equation of the separatrix is then

$$U^4 - 2U^2 + \left[1 - 2\left(\frac{1+k}{1-k}\right)V^2 - V^4\right] = 0 \tag{14}$$

and, solving the quadratic in U^2

$$U = \pm\left[1 \pm V\left(V^2 + 2\,\frac{1+k}{1-k}\right)^{1/2}\right]^{1/2} \tag{15}$$

So, we can finally exhibit the separatrix. A number of cases for various values of the parameter k are plotted in Figure 2. The curves approach $V = \pm U$ asymptotically, but the approach is less rapid the larger the value of k. Note that $V < 0$ on the outgoing separatrix for $U > 0$.

B. Area of Stable Phase Space

The stable area, A, within the separatrices of Figure 2 is

$$A = u_s^2 \int U \, dV$$

The integral can be carried out by transforming to polar coordinates with the result

$$A = u_s^2 \frac{k}{1-k} g(k)$$

$$g(k) \equiv \frac{1}{k}\left[\ln \frac{1 + \frac{1}{\sqrt{2}}(1-k^2)^{1/2}}{1 - \frac{1}{\sqrt{2}}(1-k^2)^{1/2}} - 2k \sin^{-1}\left(\frac{1-k^2}{1+k^2}\right)^{1/2} \right] \quad (16)$$

But

$$u_s^2 = -\frac{(Q + 4\pi\delta)}{E} = \frac{Q}{E} \frac{(1-k)}{k}$$

and $A = \beta_s \mathcal{E}$ for emittance $\mathcal{E}$. Thus

$$\frac{2\beta_0 \mathcal{E}}{R x_{max}^2} = g(k) \quad (17)$$

where R is the parameter defined in (3). With a choice of R and a given emittance, k is determined by (17). If we desire that the quadrupoles dominate in developing step size, then k tends to be large - for example, if R=1 and $\mathcal{E} = 0.04\pi$ mm-mrad, then k=0.88.

C. The Step Size Condition

So far, we have not determined any of the quantities Q, E and δ. We have selected a ratio R and determined a related parameter k. We also know u_o:

$$u_o^2 = x_{max}^2 \, \frac{R}{2} \, \frac{(1-k)}{k} \tag{18}$$

and therefore $u_s = (\beta_s/\beta_o)^{\frac{1}{2}} u_o$ as well. We need the step size condition to complete the process.

Consider the projection on the V-axis. Remember $V < 0$ for the outgoing particles. From (12) and (15)

$$\frac{1}{1-k}\,\frac{1}{2\pi\delta}\,\frac{dV}{dn} = -\left[1 - V\left(V^2 + 2\frac{1+k}{1-k}\right)^{1/2}\right]^{1/2} V\left[V^2 + 2\frac{1+k}{1-k}\right]^{1/2}$$

or

$$\int_{V_1}^{V_2} \frac{dV}{\left[1 - V\left(V^2 + 2\frac{1+k}{1-k}\right)^{1/2}\right]^{1/2} V\left[V^2 + 2\frac{1+k}{1-k}\right]^{1/2}} = -2\pi\delta(1-k)\int dn = -4\pi\delta(1-k)$$

where V_1 is the projection of the septum position x_s, V_2 is the projection of $x_s + \Delta$, and the integral on the right has been evaluated for two turns. The integral on the left can be performed by a change of variable of the form $V = -\left[2\frac{1+k}{1-k}\right]^{1/2}\tan\chi$. The result is

$$\ln\left\{\frac{\left[2\frac{1+k}{1-k}\right]^{1/2} U + \left[V^2 + 2\frac{1+k}{1-k}\right]^{1/2}}{-V} + \frac{1+k}{1-k}\right\}\Bigg|_{V_1}^{V_2} = \left[2\frac{1+k}{1-k}\right]^{1/2} 4\pi\delta(1-k) \tag{19}$$

It remains to specify V_1 and V_2. The geometry appears in Figure 3. The axes X, Y are x, y in units of u_s. The point X_1 corresponds to x_s and X_2 to $x_s + \Delta$. The angle θ plays the same role as before. A bit of algebra yields a closed form for V_2:

$$V_2 = -\frac{\left(\frac{X_2}{\cos\theta}\right)^2 - 1}{\left[2\left(\frac{X_2}{\cos\theta}\right)^2 + 2\frac{1+k}{1-k} - 2\right]^{1/2}} \tag{20}$$

One isn't so fortunate in relating V_1 and X_1. In terms of the angle α shown in the figure, we want the intersection of the line

$$U = V\cot\alpha + \frac{X_1}{\sin\alpha} \tag{21}$$

with the separatrix. The angle α can be found from

$$\cos(\theta+\alpha) = -\frac{V_2}{\left(\frac{X_2}{\cos\theta}\right)} \tag{22}$$

but then one must look for a root of

$$\left(V\cot\alpha + \frac{X_1}{\sin\alpha}\right) - \left[1 - V\left(V^2 + 2\,\frac{1+k}{1-k}\right)^{1/2}\right]^{1/2} = 0 \tag{23}$$

to find V_1.

D. Inefficiency

To calculate the inefficiency, we use the form

$$\epsilon = \frac{w}{2}\,\frac{1}{(dx/dn)_{x_s}} \tag{24}$$

Again, project the motion on the v-axis:

$$\left(\frac{dx}{dn}\right)_{x_s} = \left(\frac{dv}{dn}\right)_{v_1}\left(\frac{dx}{dv}\right)_{v_1} = u_s\left(\frac{dV}{dn}\right)_{V_1}\left(\frac{dX}{dV}\right)_{V_1} \tag{25}$$

In (25), the geometrical derivative, (dX/dV), is with the aid of Figure 3:

$$\left(\frac{dX}{dV}\right)_{V_1} = -\frac{\cos\left[\alpha + \tan^{-1}\left(\frac{dU}{dV}\right)_{V_1}\right]}{\cos\left[\tan^{-1}\left(\frac{dU}{dV}\right)_{V_1}\right]} \tag{26}$$

where (dU/dV) follows from (15)

$$\left(\frac{dU}{dV}\right)_{V_1} = -\frac{1}{2U_1}\left[\left(V_1^2 + 2\,\frac{1+k}{1-k}\right)^{1/2} + \frac{V_1^2}{\left(V_1^2 + 2\,\frac{1+k}{1-k}\right)^{1/2}}\right]\;;\quad U_1 \equiv U(V_1) \tag{27}$$

The other derivative in (25) is the equation of motion, evaluated at U_1, V_1; from (12)

$$\left(\frac{dV}{dn}\right)_{V_1} = 2\pi\delta(1-k)\left[-U_1 + U_1^{\,3}\right] \tag{28}$$

So the inefficiency is

$$\epsilon = \frac{w}{2u_s}\;\frac{1}{\left(\frac{dV}{dn}\right)_{V_1}\left(\frac{dX}{dV}\right)_{V_1}} \tag{29}$$

with the derivatives given by (26) and (28).

E. An Example

At this point a numerical example may be useful, if only to demonstrate that a path can be found through the equations. As in the third-integer examples, take

x_{max} = 20 mm
$\mathcal{E}$ = 0.04π mm-mrad
β_s = 225 m, β_o = 100 m
θ = 30^o
w = 0.1 mm
Δ = 10 mm
x_s = 16 mm

Then

from (17)	$k = 0.932088$
from (18)	$u_o = 5.7731$ mm
	$u_s = 8.6597$ mm
and so	$X_1 = x_s/u_s = 1.8454$
	$X_2 = (x_s+\Delta)/u_s = 3.0002$
from (20)	$V_2 = -1.2965$
from (22)	$\alpha = 38.023^o$
from (23)	$V_1 = -0.5774$
from (19)	$\delta = -0.0440$
from (9)	$Q = 0.5105$
from (3)	$E_o = 1.276\text{x}10^{-3}/\text{mm}^2$
from (29)	$\epsilon = 1.67\%$

F. The Zero Stable Phase Space Limit

The zero phase space limit for half-integer extraction is not unique - it depends on how one gets there. However, one would like the outgoing separatrices for various stable phase space areas to lie near one another. This can be accomplished by varying Q (or δ) so $k \to 1$.

From the foregoing, it should not be surprising that the expressions remain complicated in the limit. So we will only set down here the infinitesimal stable phase space separatrix and the inefficiency associated with it.

Let k_o be the initial value of k at the outset of extraction. The coordinates U, V will still be defined in terms of the initial value of u_s. Then the equations of motion become

$$\frac{1}{2\pi\delta}\frac{1}{(1-k_o)}\frac{dU}{dn} = \frac{1+k}{1-k_o}V + V^3 \implies \frac{2}{1-k_o}V + V^3 \qquad (30)$$

$$\frac{1}{2\pi\sigma}\frac{1}{(1-k_o)}\frac{dV}{dn} = -\frac{1-k}{1-k_o}U + U^3 \implies U^3 \qquad (31)$$

where the limiting case k=1 appears at the far right. Integrating the above, the trajectory through the origin is

$$U^4 = V^4 + \frac{4}{1-k_o}V^2 \qquad (32)$$

The dotted line in Figure 3 illustrates the zero stable phase space separatrix; it indeed lies close to the initial trajectory.

To calculate the inefficiency, we first need the angle α. The expression analogous to (22) is

$$\cos(\theta+\alpha) = \frac{X_2/\cos\theta}{\left[2\left(\frac{X_2}{\cos\theta}\right)^2 + \frac{4}{1-k_o}\right]^{1/2}} \qquad (33)$$

Having α, one finds V_1 as a root of

$$\left(V\cot\alpha + \frac{X_1}{\sin\alpha}\right) - \left(V^4 + \frac{4}{1-k_o}V^2\right)^{1/4} = 0 \qquad (34)$$

Equation (26) is unchanged; (27) is replaced by

$$\left(\frac{dU}{dV}\right)_{V_1} = \frac{V_1}{U_1^3}\left[V_1^2 + \frac{2}{1-k_0}\right] \tag{35}$$

and (28) changes to

$$\left(\frac{dV}{dn}\right)_{V_1} = 2\pi\delta(1-k_0)U_1^3 \tag{36}$$

If we continue the example of Section E:

from (33) $\alpha = 36.587^\circ$
from (34) $V_1 = -0.6669$
from (32) $U_1 = 2.1977$
from (35) $dU/dV = -1.6617$
from (26) $dX/dV = -1.7934$
from (36) $dV/dn = -0.2258$
from (29) $\epsilon = 1.43\%$

V. Discussion

Let us begin this discussion by looking at Figure 4, where various quantities associated with half-integer extraction are shown as a function of the parameter R. Recall that R expresses the degree to which quadrupoles dominate in producing step size. After all the involved development of Section IV, Figure 4 is something of an anti-climax. Though the tune shift, δ, and the quadrupole and octupole strengths vary extensively, the inefficiency stays close to its minimum value of 1.63% over most of the range in R shown. Only for $R < 0.5$ does the inherent lower efficiency of the octupoles begin to appear.

The half-integer process is a shade more efficient than third-integer for the cases considered here, but the difference - 1.63% versus 1.69% for an emittance of 0.04π mm-mrad - is hardly significant. However, it is significant that the two extraction modes cannot be distinguished on the basis of efficiencies alone.

In third-integer extraction, zero stable phase space is obtained for the tune exactly at the resonant value of 19+1/3. In contrast, the stable phase space vanishes in the half-integer case for sufficiently large stop-band width - any $k > 1$ - as the linear motion becomes unstable. That is, the half-integer process is better equipped to clear all of the beam out of the accelerator in the face of magnet ripple and tune spreads.

One should take into account the dominant multipole error fields in the main magnets. At the time (mid-1972) that the extraction mode for the main ring was chosen, the main ring had rather large high-field sextupole terms the distribution of which was unknown. On the other hand, there was a zero harmonic octupole term of sign and magnitude suitable for the half-integer method. It is too early to make the corresponding analysis for the Doubler, though the present sextupole fields of the dipoles are cause for concern.

As mentioned in the introduction, the purpose of this memorandum is to present parallel treatments of the two processes without refinement beyond the point reached here. If it were necessary to select a process today, the results thus far would favor the half-integer technique. Fortunately, one needn't choose today; it may be possible to go on to the refinements.

For example, no mention has been made as yet of the tune and position spreads of the beam due to momentum width. Chromaticity control will be available to adjust the former, and, in principle at least, the momentum dispersion function could be manipulated if sustantial benefits were to be found. It would be useful to obtain a better insight into the chromaticity adjustment range needed than we have at present.

The emittance of the extracted beam is another subject deserving study. Note that the elongation of the stable phase space due to the quadrupole harmonic in half-integer suggests that a substantial reduction of the outgoing emittance can be achieved. It will be interesting to see if this effect survives after the inclusion of tune and position spreads.

Finally, the relative merits of the two extraction techniques for fast resonant beam spill must be examined. For removal of all the beam in a single burst, the half-integer resonance has the advantage noted above that the linear motion is made unstable. However, attempts thus far to extract more than one burst from the main ring have been only partially successful. Needless to say, the capability of extraction several ~1 msec pulses on flat-top would be very valuable.

References

1. See, for example, L. C. Teng, in "The Energy Doubler", 1976
2. T. L. Collins, to appear in Energy Doubler Design Progress Report, 1979
3. D. A. Edwards, ibid.
4. T. L. Collins, private communication
5. H. T. Edwards and M. Harrison, to appear in Energy Doubler Design Progress Report, 1979

APPENDIX

For completeness, we outline below one method of obtaining the equations of motion. Only the lowest order terms in the perturbing fields will be included.

Before any of the extraction devices are turned on, a betatron oscillation in the horizontal plane is described by

$$x = a\left(\frac{\beta(z)}{\beta_0}\right)^{1/2}\cos\chi(z) \tag{1}$$

where x is the displacement from the central trajectory, z, the independent variable, is a coordinate along the direction of that trajectory, β (z) is the amplitude function, β_0 is its value at some point of interest, and χ (z) is the phase. The amplitude a is an invariant in the unperturbed machine.

Differentiation of (1) gives the slope x':

$$x' \equiv \frac{dx}{dz} = -\frac{a}{[\beta_0\beta(z)]^{1/2}}\sin\chi - \frac{a\,\alpha(z)}{[\beta_0\beta(z)]^{1/2}}\cos\chi \tag{2}$$

Rather than using x' itself, it is convenient to define a variable y by

$$y \equiv \beta(z)x' + \alpha(z)x \tag{3}$$

which, from (2), is

$$y = -a\left(\frac{\beta(z)}{\beta_0}\right)^{1/2}\sin\chi \tag{4}$$

Assume a magnetic field B(x,z), perpendicular to x and z, is introduced at z and extends over a length Δz. The sign of B is positive if B is directed in the same sense as the guide field. For sufficiently small Δz, x does not change as the particle traverses Δz, but

$$\Delta x' = -\frac{B\,\Delta z}{(B\rho)} \tag{5}$$

and so

$$\Delta y = -\beta(z)\frac{B\,\Delta z}{(B\rho)} \tag{6}$$

As a result of the perturbation, the amplitude a and phase have changed. From

$$\Delta x = \left(\frac{\beta(z)}{\beta_0}\right)^{1/2}\left[\Delta a\cdot\cos\chi - a\sin\chi\cdot\Delta\chi\right] = 0 \tag{7}$$

$$\Delta y = -\left(\frac{\beta(z)}{\beta_0}\right)^{1/2}\left[\Delta a\cdot\sin\chi + a\cos\chi\cdot\Delta\chi\right] = -\beta(z)\frac{B\,\Delta z}{(B\rho)} \tag{8}$$

one has

$$\Delta a = \frac{\beta_0}{(B\rho)}\left(\frac{\beta(z)}{\beta_0}\right)^{1/2} B\,\Delta z\,\sin\chi \tag{9}$$

$$\Delta\chi = \frac{\beta_0}{(B\rho)}\left(\frac{\beta(z)}{\beta_0}\right)^{1/2}\frac{B\,\Delta z}{a}\cos\chi \tag{10}$$

Suppose that we are observing successive passages of the particle at some point of the ring where $\beta = \beta_0$. If the phase at a given passage is ψ, then on the succeeding turn, the unperturbed phase χ would develope according to

$$\chi(z) = \psi + \nu\varphi(z)\;;\quad \varphi(z) \equiv \int\frac{dz}{\nu\beta(z)} \tag{11}$$

To obtain the first order equations of motion, we assume that the changes in a and ψ due to the perturbing fields can be found by adding up the contributions (9) and (10) as though the motion were unperturbed in evaluating B(x,z) and χ over one turn; i.e.,

$$\frac{da}{dn} = \frac{\beta_0}{(B\rho)}\oint dz\left(\frac{\beta(z)}{\beta_0}\right)^{1/2} B(x,z)\sin\left[\psi + \nu\varphi(z)\right] \tag{12}$$

$$\frac{d}{dn}(\psi - 2\pi\nu n) = \frac{\beta_0}{(B\rho)}\oint dz\left(\frac{\beta(z)}{\beta_0}\right)^{1/2}\frac{B(x,z)}{a}\cos\left[\psi + \nu\varphi(z)\right] \tag{13}$$

Now, consider a sextupole perturbation: B(x,z) of the form

$$B(x,z) = \frac{B''}{2} x^2 \tag{14}$$

Insertion of (14) into (12) with x expressed as in (1) yields, after reduction of the trig functions,

$$\frac{da}{dn} = \frac{1}{4} a^2 \frac{\beta_0}{(B\rho)} \oint dz \left(\frac{B''}{2}\right) \left[\sin\psi \cos\nu\varphi + \cos\psi \sin\nu\varphi + \sin 3\psi \cos 3\nu\varphi + \cos 3\psi \sin 3\nu\varphi \right] \left(\frac{\beta(z)}{\beta_0}\right)^{3/2} \tag{15}$$

If the tune were close to an integer, the first two terms in the integrand of (15) would be of interest. But for ν not near an integer, $\sin\psi$ and $\cos\psi$ will change rapidly from turn to turn and so the amplitude will not grow steadily. However, if 3ν were an integer, $\sin 3\psi$ and $\cos 3\psi$ would have constant values from turn to turn, and then the amplitude would change steadily. So we can ignore the first two terms and retain the second pair. Since we want to study the case where 3ν is not exactly an integer, let $3\nu_0$ denote the integer of interest, with the difference $\delta \equiv \nu - \nu_0$ small compared with unity. Equation (15) becomes

$$\frac{da}{dn} = \frac{1}{4} a^2 \left[S_1 \sin 3\psi + S_2 \cos 3\psi \right] \tag{16}$$

where

$$\begin{aligned} S_1 &\equiv \frac{\beta_0}{(B\rho)} \oint dz \left(\frac{B''}{2}\right) \left(\frac{\beta(z)}{\beta_0}\right)^{3/2} \cos 3\nu_0\varphi \\ S_2 &= \frac{\beta_0}{(B\rho)} \oint dz \left(\frac{B''}{2}\right) \left(\frac{\beta(z)}{\beta_0}\right)^{3/2} \sin 3\nu_0\varphi \end{aligned} \tag{17}$$

In defining S_1 and S_2 by (17), we have used the proximity of ν to ν_0 so that S_1 and S_2 are true harmonic amplitudes.

The equation of motion for ψ is found by the same procedure, but with one modification. The phase itself advances by $2\pi\nu$ in one turn, and so ψ hardly qualifies as a continuous variable.

This circumstance was already recognized in writing the left hand side of (13). Now observe that the ψ-related factors that enter the right hand sides of the equations, $\cos 3\psi$ and $\sin 3\psi$, are insensitive to the replacement of ψ by $\psi - 2\pi\nu_0 n$. We change the variable accordingly, and obtain the equation of motion for the new phase:

$$\frac{d\psi}{dn} = \frac{1}{4} a \left[S_1 \cos 3\psi - S_2 \sin 3\psi \right] + 2\pi\delta \tag{18}$$

With the foregoing redefinition, ψ becomes a variable continuous in n, and can be used as a polar angle in representing phase space trajectories.

The equations of motion were developed in the phase, amplitude form because the characteristic of a resonance - amplitude growth - is more readily identified thereby. Transformation of the equations of motion to x, y coordinates follows from

$$\frac{dx}{dn} = \frac{x}{a}\left(\frac{da}{dn}\right) + y\left(\frac{d\psi}{dn}\right) \tag{19}$$

$$\frac{dy}{dn} = \frac{y}{a}\left(\frac{da}{dn}\right) - x\left(\frac{d\psi}{dn}\right) \tag{20}$$

and are used in the latter form in the body of the text.

The procedure for the half-integer case is identical. The algebra is a bit more lengthy, since both quadrupole and octupole terms must be included.

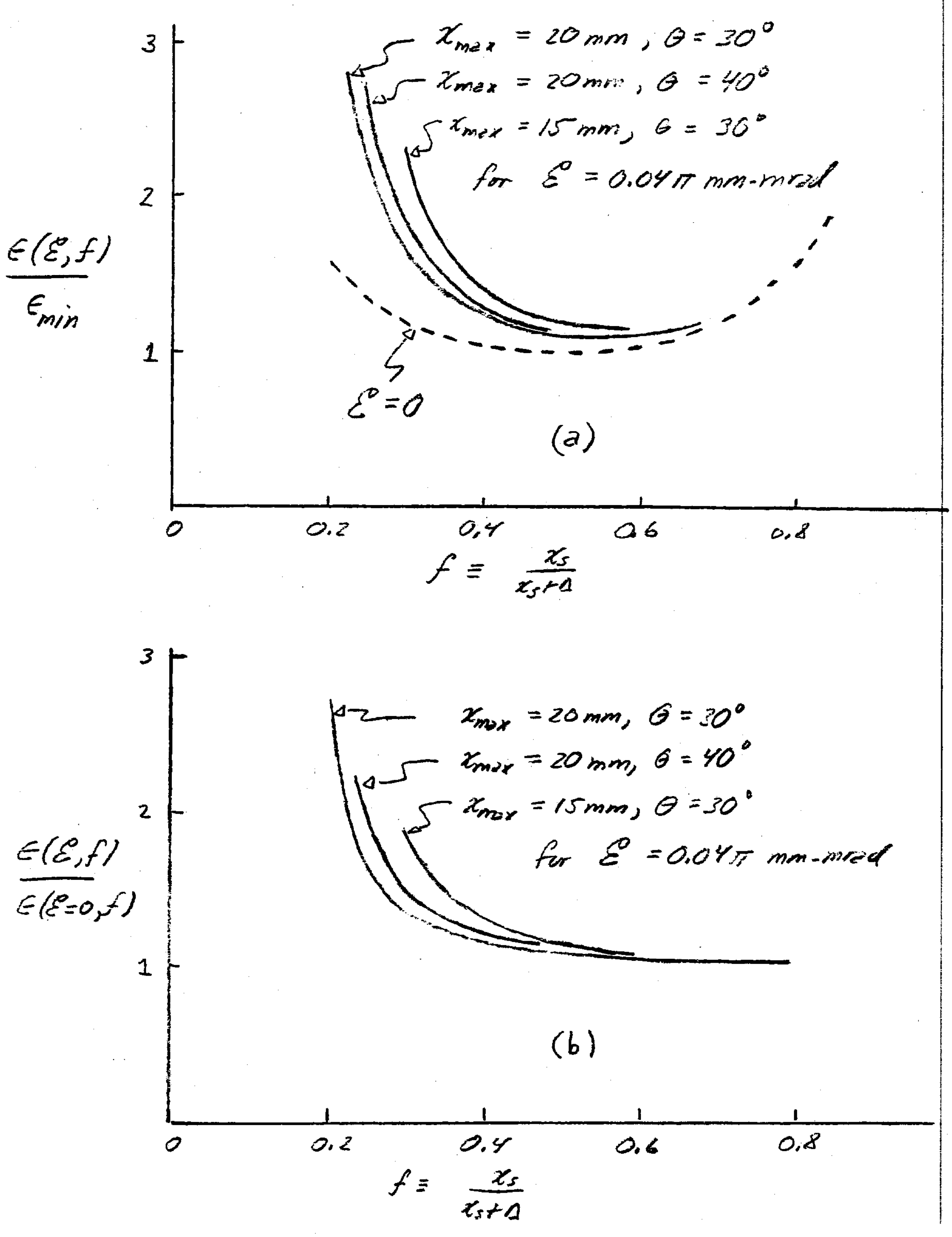


Figure 1. The inefficiency for third-integer extraction is shown in (a) relative to the minimum inefficiency for zero stable phase space, and in (b), relative to the inefficiency for zero stable phase space at the same value of f.

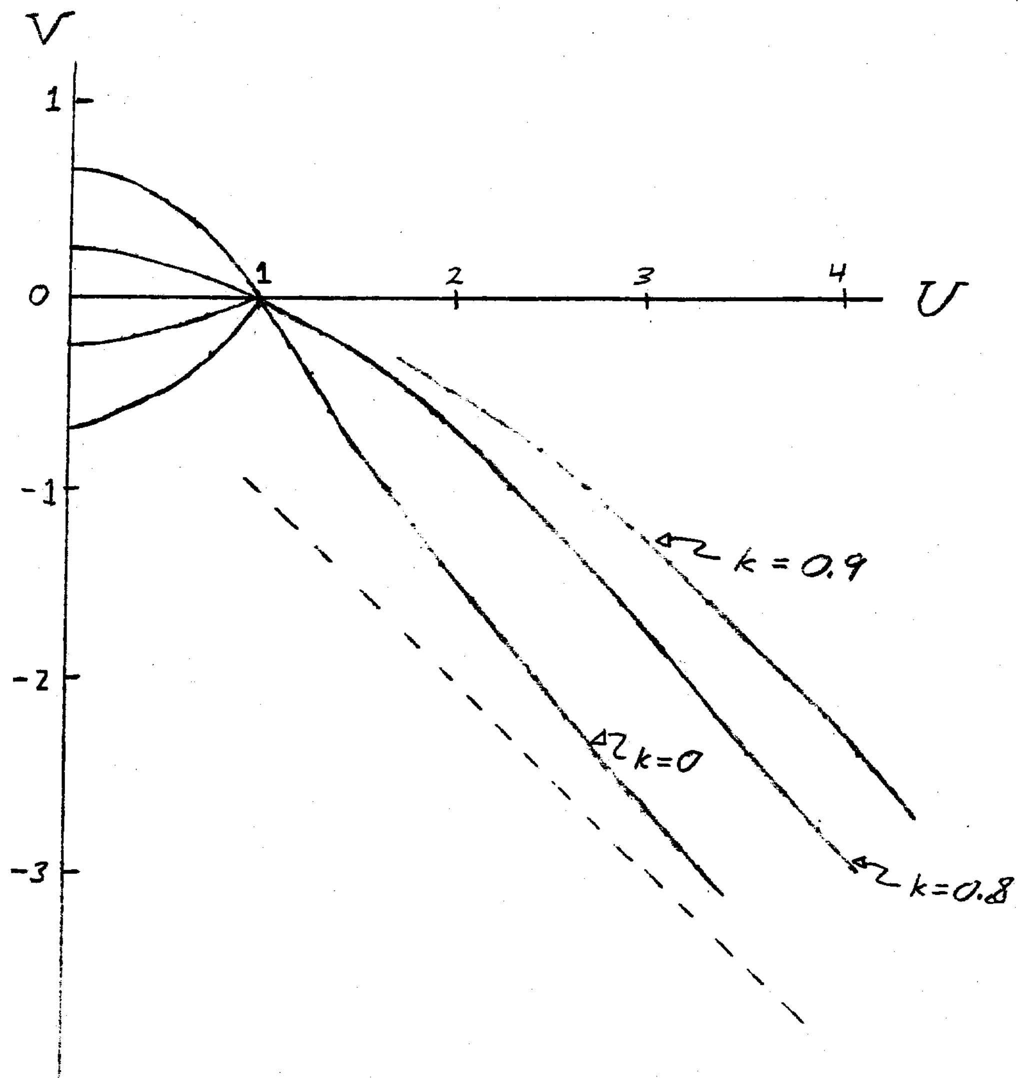


Figure 2. A few examples of the separatrices for half-integer extraction are presented here. Although the outgoing trajectories approach the line V=-U (shown dotted), the approach for the large values of the parameter k, the cases of interes is quite gradual.

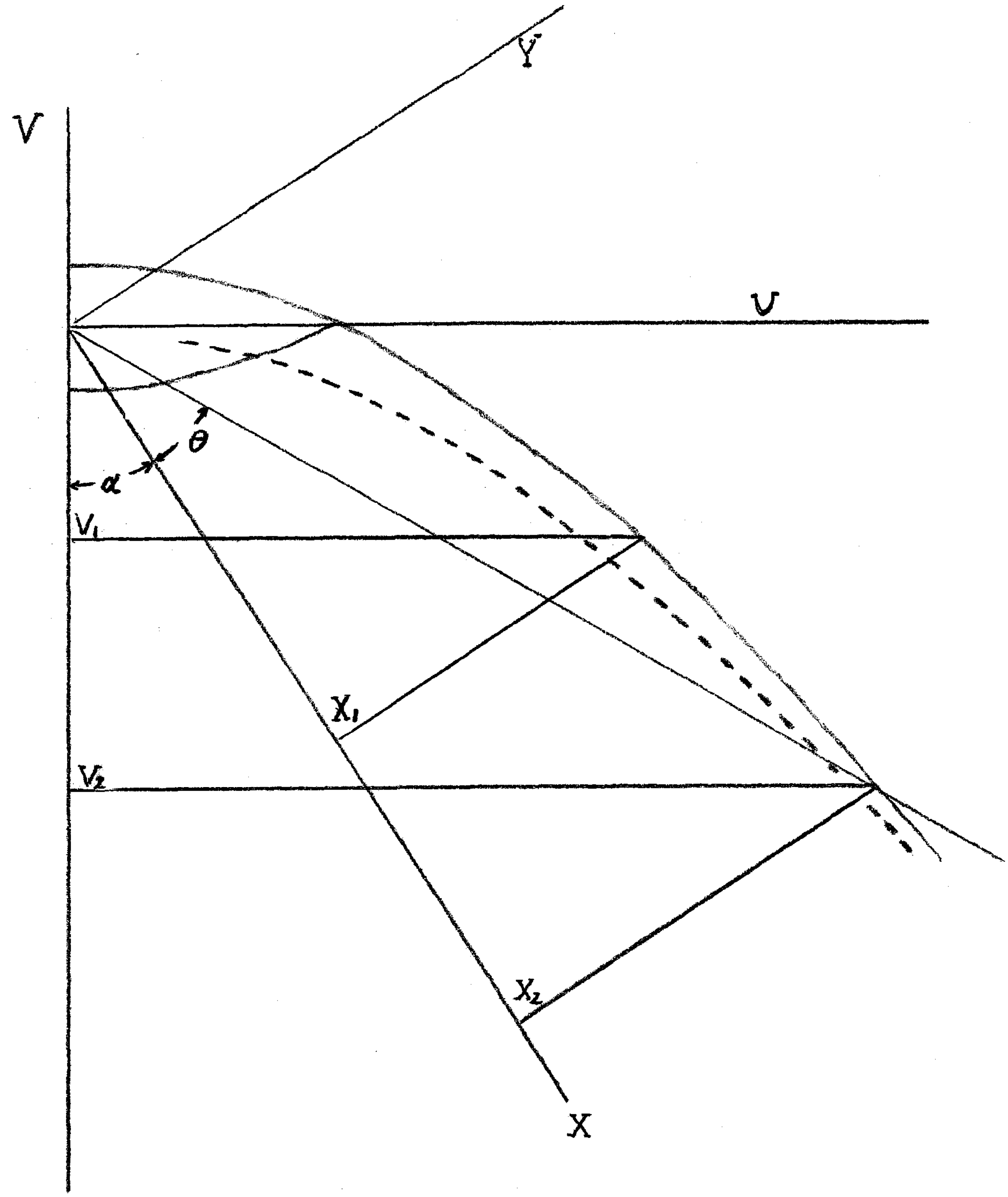
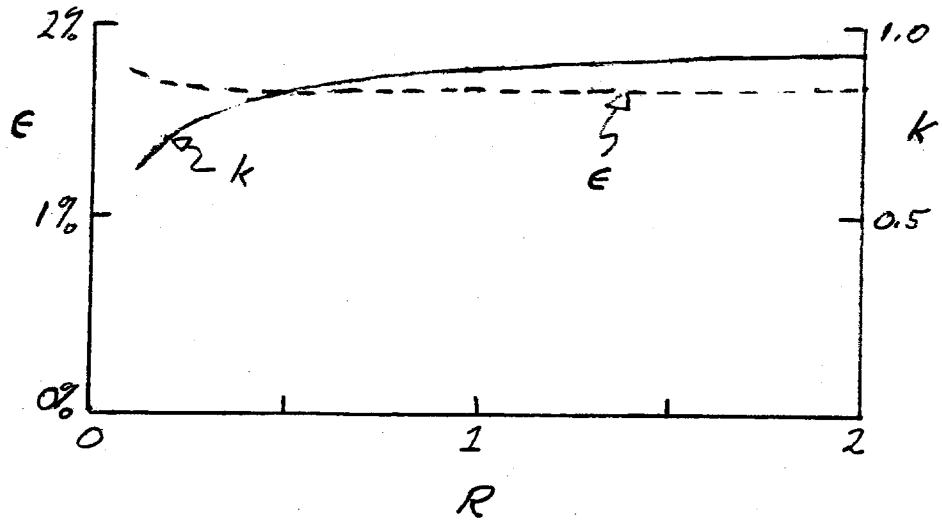


Figure 3. The relationship between quantities on the X and V axes is illustrated. The dotted line is the separatrix for vanishing stable phase space.



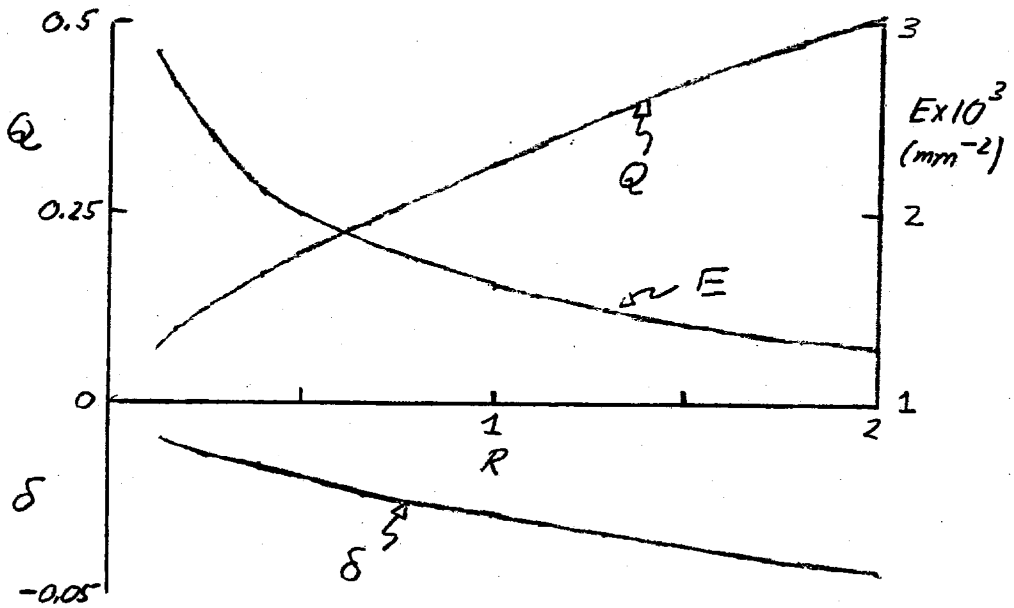


Figure 4. The dependence of various parameters in half-integer extraction as a function of the ratio R is illustrated by the plots above.